A few properties of the eigenvalues of normalized graph Laplacian

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Abstract

Here we have investigated a few properties of the eigenvalues of normalized (geometric) graph Laplacian in different graphs. Preservation of eigenvalue 1 from a particular subgraph to the entire graph, the spectrum of the graph constructed with triangles share a common vertex have been addressed. Further using the number and degrees of common neighbors between vertices some new upper bounds for the largest eigenvalue have been introduced.

1 Introduction

Let $\Gamma = (V, E)$ be a finite, connected and undirected graph with the vertex set $V = \{v_1, \dots, v_n\}$. Two vertices $v_i, v_j \in \Gamma$ are called neighbor if they are connected by an edge from the set E and we denote it by $v_i \sim v_j$. If vertices $v_i, v_j \in \Gamma$ are not connected by an edge we denote it by $v_i \nsim v_j$. Let n_{v_i} be the degree, the number of neighbors, of the vertex v_i. For any functions $g : V \to \mathbb{R}$ we define the normalized graph Laplacian as

$$\Delta g(v_i) := g(v_i) - \frac{1}{n_{v_i}} \sum_{v_j; v_j \sim v_i} g(v_j). \tag{1}$$

This is different from the (algebraic) graph Laplacian operator,

$$Lg(v_i) := n_{v_i} g(v_i) - \sum_{v_j; v_j \sim v_i} g(v_j) \tag{2}$$

which is well studied in the graph theoretical literature (see [3, 5, 9, 10, 2]). But the operator (1) is similar to the Laplacian:

$$\mathcal{L}g(v_i) := g(v_i) - \sum_{v_j;v_j \sim v_i} \frac{1}{\sqrt{n_{v_i} n_{v_j}}} g(j) \tag{3}$$

investigated in [4] and thus (1) and (3) have the same spectrum.

The matrix $\Delta = [a_{ij}]$ representation of our operator in (1) is as follows:

$$a_{ij} = \begin{cases} 1, & \text{if } i = j \\ -\frac{1}{n_{v_i}}, & \text{if } v_i \sim v_j \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

2 Some basic properties

Here we recall some of the basic properties of the eigenvalues and eigenfunctions of this operator from [6, 3, 1]. For any functions $f, g : V \to \mathbb{R}$ the normalized graph Laplacian (NGL) is symmetric for the product

$$(f, g) := \sum_{v_i \in V} n_{v_i} f(i) g(i) \tag{5}$$

Since $(f, \Delta g) = (\Delta f, g)$ and $(\Delta g, g) \geq 0$ all eigenvalues of Δ is real and non-negative. The eigenvalue equation of Δ is

$$\Delta f - \lambda f = 0 \tag{6}$$

where a nonzero solution f is called an eigenfunction corresponding to the eigenvalue λ. Now if we arrange all the eigenvalues in non-decreasing manner we have

$$\lambda_0 = 0 < \lambda_1 \leq ... \leq \lambda_{n-1} \leq 2.$$

A graph is bipartite *iff* $\lambda_{n-1} = 2$. For a connected graph only the smallest eigenvalue λ_0 is 0 with an constant eigenfunction. Hence, for all other eigenfunctions f, from (5) we get

$$\sum_{v_i \in V} n_{v_i} f(v_i) = 0 \tag{7}$$

The smallest non-trivial eigenvalue λ_1 can be estimated as

$$\lambda_1 = min\{\frac{\sum_{i,j;v_i \sim v_j} (f(v_i) - f(v_j))^2}{\sum_i n_{v_i} f(v_i)^2} : \sum_i n_{v_i} f(v_i) = 0\} \tag{8}$$

where as the highest eigenvalue λ_{n-1} which bounded above by 2 (i.e. $\lambda_{n-1} \leq 2$) can be estimated as

$$\lambda_{n-1} = max\{\frac{\sum_{i,j;v_i \sim v_j} (f(v_i) - f(v_j))^2}{\sum_i n_{v_i} f(v_i)^2} : \sum_i n_{v_i} f(v_i) = 0\} \tag{9}$$

3 Some new results

Theorem 3.1. *Let Γ be a graph consists of only m triangles share a common vertex. Then the eigenvalues of Γ are: 0, 1/2 with multiplicity $m-1$, and 3/2 with multiplicity $m+1$.*

Proof. Let $V = \{v_1, v_2, \ldots, v_{2m+1}\}$ be the vertex set of the graph Γ and the triangles in Γ are formed with the vertices:
$\{v_1, v_2, v_{2m+1}\}, \{v_3, v_4, v_{2m+1}\}, \ldots, \{v_{2m-1}, v_{2m}, v_{2m+1}\}$.

Now using the equations (5), (7), (8), and (9) we construct the eigenvectors $X_1^{1/2}, X_2^{1/2}, \ldots, X_{m-1}^{1/2}$ and $X_1^{3/2}, X_2^{3/2}, \ldots, X_{m+1}^{3/2}$ corresponding to the eigenvalues 1/2 and 3/2 respectively as follows:

$$X_1^{1/2} = (\underbrace{1, 1, -1, -1}_{2+2.1}, 0, \ldots, 0)^T,$$

$$X_2^{1/2} = (\underbrace{1, 1, 1, 1, -2, -2}_{2+2.2}, 0, \ldots, 0)^T,$$

$$\vdots$$

$$X_{m-1}^{1/2} = (\underbrace{1, 1, 1, 1, \ldots, 1, -(m-1), -(m-1)}_{2+2.(m-1)}, 0)^T$$

and

$$X_1^{3/2} = (\underbrace{1, -1}_{2.1}, 0, 0, 0, \ldots, 0)^T,$$

$$X_2^{3/2} = (\underbrace{0, 0, 1, -1}_{2.2}, 0, \ldots, 0)^T,$$

$$\vdots$$

$$X_m^{3/2} = (\underbrace{0, \ldots, 0, 1, -1}_{2.m}, 0)^T,$$

$$X_{m+1}^{3/2} = (\underbrace{1, \ldots, 1}_{2.m}, -2)^T.$$

Hence the proof $\square$

Theorem 3.2. *Let $\Gamma = (V, E)$ be a graph and $V' = \{v_1, v_2, \ldots, v_r\}$ be a subset of V having the same set of neighbors $V'' = \{v_{r+1}, v_{r+2}, \ldots, v_{r+m}\}$, where $V = \{v_1, \ldots, v_r, v_{r+1}, \ldots, v_{r+m}, v_{r+m+1} \ldots, v_n\}$. Let construct the graphs $\Gamma' = (V', E')$ where $E' \subseteq V' \times V'$ and $\Gamma^+ = (V, E^+)$ where $E^+ = E \cup E'$. Now for each eigenfunction f' corresponding to eigenvalue 1 of Γ' with $\sum_{i=1}^{r} f'(v_i) = 0$, there exists an eigenfunction f^+ corresponding to eigenvalue 1 of Γ^+ where*

$$f^+ = \begin{cases} f'(v_i), & i = 1, \dots, r \\ 0, & i > r \end{cases}$$

Proof. Let $V_0 = \{v_{r+m+1}, \dots, v_n\}$. Now $V = V' \cup V'' \cup V_0$.
Since f' is an eigenfunction of Γ' corresponding to the eigenvalue 1,

$$1.n'_{v_i} f'(v_i) = n_{v_i} f'(v_i) - \sum_{v_j; v_j \sim v_i, v_i v_j \in E'} f'(v_j), \forall i = 1, \dots, r \tag{10}$$

where n'_{v_i} is the degree of v_i in Γ'. Now adding both side of $mf'(x_i)$ we get

$$1.(m + n'_{v_i}) f'(v_i) = (m + n'_{v_i}) f'(v_i) - \sum_{v_j; v_j \sim v_i, v_i v_j \in E'} f'(v_j), \forall i = 1, \dots, r \tag{11}$$

Since $f^+(v_j) = 0$ if $v_j \notin V'$ the equation (11) implies

$$1.n^+_{v_i} f^+(v_i) = n^+_{v_i} f^+(v_i) - \sum_{v_j; v_j \sim v_i, v_j \in V'} f^+(v_j), \forall i = 1, \dots, r \tag{12}$$

where $n^+_{v_i}$ is the degree of $v_i \in \Gamma^+$. Now using $\sum_{i=1}^r f'(v_i) = 0 \Rightarrow \sum_{i=1}^r f^+(v_i) = 0$ and $f^+(v_k) = 0 \forall v_k \notin V'$ we get

$$1.n^+_{v_{r+i}} f^+(v_{r+i}) = n^+_{v_{r+i}} f^+(v_{r+i}) - \sum_{j=1}^{r} f^+(x_j) - \sum_{v_k; v_k \sim v_{r+i}, v_k \in V_0 \cup V''} f^+(v_k), \forall i = 1, \dots, m. \tag{13}$$

and

$$1.n^+_{v_{r+m+i}} f^+(v_{r+m+i}) = n^+_{v_{r+m+i}} f^+(v_{r+m+i}) - \sum_{v_j; v_j \sim v_{r+m+i}, v_j \in V''} f^+(v_j), \forall i = 1, \dots, n-(r+m) \tag{14}$$

The equations (12), (13), and (14) imply that f^+ is an eigenfunction of Γ' corresponding eigenvalue 1. □

A similar result can be found in [8] for non-normalized (algebraic) graph Laplacian.

Theorem 3.3. *For a connected graph Γ, let λ is any nontrivial eigenvalue. Then*

$$\lambda \le 2 - \min_{v_i; v_j \sim v_i} \frac{|N_{v_i v_j}|}{n_{v_i}}$$

where $|N_{v_i v_j}|$ is the number of common neighbor between vertex v_i and v_j.

Proof. Let f is an eigenfunction for a nontrivial eigenvalue λ.
We can assume that for a vertex v_i, $f(v_i) = 1$ and $|f(v_k)| \le 1 \forall k \ne i$. Let us choose a vertex v_j, a neighbor of v_i, such that $f(v_j) = \min_k \{f(v_k) : v_k \sim v_i\}$.

Now the eigenvalue equation is

$$\lambda f(v_l) = f(v_l) - \frac{1}{n_{v_l}} \sum_{v_k; v_k \sim v_l} f(v_k) \text{ for all } l. \tag{15}$$

By putting $l = i$ and $l = j$ in (15) and subtracting we get

$$\begin{aligned} \lambda(1 - f(v_j)) &= (1 - f(v_j)) + (\frac{1}{n_{v_j}} - \frac{1}{n_{v_i}}) \sum_{v_k; v_k \sim v_i, v_k \sim v_j} f(v_k) + \frac{1}{n_{v_j}} \sum_{v_k; v_k \sim v_j, v_k \nsim v_i} f(v_k) - \\ &\quad - \frac{1}{n_{v_i}} \sum_{v_k; v_k \sim v_i, v_k \nsim v_j} f(v_k), \qquad \text{since } f(v_i) = 1. \end{aligned} \tag{16}$$

That implies

$$\lambda(1-f(v_j)) \leq (1-f(v_j)) + (\frac{1}{n_{v_j}} - \frac{1}{n_{v_i}})|N_{v_i v_j}| + \frac{1}{n_{v_j}}(n_{v_j} - |N_{v_i v_j}|) - \frac{1}{n_{v_i}}(n_{v_i} - |N_{v_i v_j}|) f(v_j) \tag{17}$$

That implies

$$\lambda(1 - f(v_j)) \leq (1 - f(v_j))(2 - \frac{|N_{v_i v_j}|}{n_{v_i}}) \tag{18}$$

If $f(v_j) = 1$, then $f(v_k) = 1$ for all $v_k \sim v_i \Rightarrow \lambda = 0$, from (15)

Hence the proof

□

Corollary 3.1. *For a connected K-regular graph Γ, let λ is any nontrivial eigenvalue. Then*

$$\lambda \leq 2\sqrt{1 - \frac{N_i'}{2K^2}}$$

where $N_i' = \min_{v_i}\{\sum_{v_j \sim v_i} |N_{v_i v_j}|\}$ and $|N_{v_i v_j}|$ is the number of common neighbor between vertex v_i and v_j.

Proof. Let f is an eigenfunction for a nontrivial eigenvalue λ.
We can assume that for a vertex v_i, $f(v_i) = 1$ and $|f(v_k)| \leq 1 \ \forall k$. Let v_j be a neighbor of v_i.

Since $f(v_i) = 1$ the eigenvalue equation become

$$K\lambda = K - \sum_{v_j; v_j \sim v_i} f(v_j) \tag{19}$$

Now since $f(v_k) \geq -1 \ \forall k$

$$\sum_{v_k; v_k \sim v_i, v_k \nsim v_j} f(v_k) \geq -(n_{v_i} - |N_{v_i v_j}|) \tag{20}$$

and since $f(v_k) \leq 1\ \forall k$

$$\sum_{v_k; v_k \sim v_j, v_k \nsim v_i} f(v_k) \geq -(n_{v_j} - |N_{v_i v_j}|) \tag{21}$$

Now take $n_{v_i} = n_{v_k} = K$ in equation (17) in Theorem (3.3) we get

$$K\lambda(1 - f(v_j)) = K(1 - f(v_j)) + \sum_{v_k; v_k \sim v_j, v_k \nsim v_i} f(v_k) - \sum_{v_k; v_k \sim v_i, v_k \nsim v_j} f(v_k) \tag{22}$$

Using the equation (20) and (21) we get

$$K\lambda(1 - f(v_j)) \leq 2(2K - |N_{v_i v_j}|) \tag{23}$$

Taking summation over j, such that $v_j \sim v_i$, on both side of (23) and using (19) we get

$$K^2\lambda^2 \leq 2(2K^2 - \sum_{v_j; v_j \sim v_i} |N_{v_i v_j}|) \leq 4K^2(1 - \frac{1}{2K^2} \min_{v_i} \sum_{v_j; v_j \sim v_i} |N_{v_i v_j}|) \tag{24}$$

Hence the proof

□

Theorem 3.4. *For a connected graph Γ, let λ is any nontrivial eigenvalue. Then*

$$\lambda \leq \max_{v_i} \left\{ \sqrt{3 + \frac{1}{n_{v_i}} \Big\{ \sum_{v_j; v_j \sim v_i} \frac{1}{n_{v_j}} + \sum_{v_j; j \neq i} \sum_{v_k; v_k \sim v_i, v_k \sim v_j} \frac{1}{n_{v_k}} \Big\}} \right\}.$$

Proof. Let $|N_{v_i v_j}|$ is the number of common neighbor between vertex v_i and v_j. Let us consider the matrix Δ^2 which is the square of (4). The (i, j)th element of the matrix form Δ^2 is:

$$\begin{cases} 1 + \frac{1}{n_{v_i}} \sum_{v_j; v_j \sim v_i} \frac{1}{n_{v_j}}, & \text{if } i = j \\ -\frac{1}{n_{v_i}}(2 - \sum_{v_k; v_k \sim v_i, v_k \sim v_j} \frac{1}{n_{v_k}}), & \text{if } v_i \sim v_j \text{ and } |N_{v_i v_j}| \neq 0 \\ -\frac{1}{n_{v_i}}(-\sum_{v_k; v_k \sim v_i, v_k \sim v_j} \frac{1}{n_{v_k}}), & \text{if } v_i \nsim v_j \text{ and } |N_{v_i v_j}| \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Using Gershgorins theorem to the row of Δ^2, we get

$$\begin{aligned} \lambda^2 &\leq \max_{v_i} \left\{ \Big(1 + \frac{1}{n_{v_i}} \sum_{v_j; v_j \sim v_i} \frac{1}{n_{v_j}}\Big) + \frac{1}{n_{v_i}} \sum_{v_j; v_j \sim v_i} \Big| 2 - \sum_{v_k; v_k \sim v_i, v_k \sim v_j} \frac{1}{n_{v_k}} \Big| + \frac{1}{n_{v_i}} \sum_{v_j; v_j \nsim v_i} \sum_{v_k; v_k \sim v_i, v_k \sim v_j} \frac{1}{n_{v_k}} \right\} \\ &\leq \max_{v_i} \Bigg\{ 1 + \frac{1}{n_{v_i}} \sum_{v_j; v_j \sim v_i} \frac{1}{n_{v_j}} + \frac{1}{n_{v_i}} \sum_{v_j; v_j \sim v_i} 2 + \frac{1}{n_{v_i}} \sum_{v_j; v_j \sim v_i} \sum_{v_k; v_k \sim v_i, v_k \sim v_j} \frac{1}{n_{v_k}} \\ &\qquad + \frac{1}{n_{v_i}} \sum_{v_j; v_j \nsim v_i} \sum_{v_k; v_k \sim v_i, v_k \sim v_j} \frac{1}{n_{v_k}} \Bigg\} \end{aligned} \tag{25}$$

That implies

$$\lambda^2 \leq \max_{v_i} \left\{ 3 + \frac{1}{n_{v_i}} \sum_{v_j; v_j \sim v_i} \frac{1}{n_{v_j}} + \frac{1}{n_{v_i}} \sum_{v_j; j \neq i} \sum_{v_k; v_k \sim v_i, v_k \sim v_j} \frac{1}{n_{v_k}} \right\} \tag{26}$$

Hence the proof ☐

Corollary 3.2. *If λ is any nontrivial eigenvalue of a connected graph Γ, which does not have any triangle. Then*

$$\lambda \leq \max_{v_i} \left\{ \sqrt{3 + \frac{1}{n_{v_i}} \Big\{ \sum_{v_j; v_j \sim v_i} \frac{1}{n_{v_j}} + \sum_{v_j; v_j \nsim v_i} \sum_{v_k; v_k \sim v_i, v_k \sim v_j} \frac{1}{n_{v_k}} \Big\}} \right\}.$$

Proof. If there is no triangle in the graph Γ, then the equation (25) becomes

$$\lambda^2 \leq \max_{v_i} \left\{ 1 + \frac{1}{n_{v_i}} \sum_{v_j; v_j \sim v_i} \frac{1}{n_{v_j}} + \frac{1}{n_{v_i}} \sum_{v_j; v_j \sim v_i} 2 + \frac{1}{n_{v_i}} \sum_{v_j; v_j \nsim v_i} \sum_{v_k; v_k \sim v_i, v_k \sim v_j} \frac{1}{n_{v_k}} \right\} \tag{27}$$

Hence the proof ☐

A similar result results in theorem (3.3) and (3.4) can be found in [7] for (algebraic) graph Laplacian.

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